Refined Large Deviation Asymptotics for the Classical Occupancy Problem*

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Abstract

In this paper refined large deviation asymptotics are derived for the classical occupancy problem. The asymptotics are established for a sequential filling experiment and an occupancy experiment. In the first case the random variable of interest is the number of balls required to fill a given fraction of the urns, while in the second a fixed number of balls are thrown and random variable is the fraction of nonempty urns.

1 Introduction

The classical occupancy problem [8] is concerned with the number of occupied urns after r balls have been thrown into n urns, and with the balls thrown according to Maxwell-Boltzmann (MB) statistics (i.e., each ball enters any urn with equal probability, and different throws are independent of one another). It is a fundamental model that appears in many contexts.

The occupancy problem can be regarded from two related points of view. One perspective focuses on the *filling* process and the other on the *occupancy* process. In the filling process, balls are thrown in an endless sequence and we

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record the number of balls that must be thrown before a previously empty urn becomes nonempty. This produces a sequence of integer valued random variables whose sum is the random number of balls needed occupy a given number of urns. In the occupancy process, a fixed number of balls are thrown and we record the fraction of occupied urns after each ball is thrown. This produces a sequence of random variables that is monotonically increasing, with the last variable representing the random fraction of occupied urns after all balls are thrown.

Clearly these two processes are closely related, in that one is essentially the inverse of the other. Indeed, it is tempting to think they give exactly the same information and that statements on the asymptotic behavior of one immediately translate into statements on the asymptotic behavior of the other. Although this would be true if the processes were strictly monotonic, in fact they are not. The differences between the information contained in the processes is made precise in Section 2.2. We will see that once these differences are accounted for, it is still possible to analyze the behavior each model once one understands the other.

For the occupancy process, process level large deviation principle (LDP) results for the fraction of occupied urns are given in [19], together with the solution to the associated calculus of variations problem when the terminal value of the trajectory is fixed (this identifies the rate function for the outcome of the occupancy experiment). The papers [7, 4] generalize the process level LDP determined in [19], and moreover [7] obtains strong minimizing extremals to the associated calculus of variations problem in this more general setting.

On the other hand, the paper [16] directly studies the filling process. Using the Gärtner-Ellis Theorem [6] and a representation for the number of balls needed to fill a given fraction of urns (the filling experiment) as a sum of random variables, [16] proves an LDP for the ratio of this number to the number of urns.

The analysis of the occupancy process, as in [7], provides greater qualitative insight into the behavior of occupancy problems. However, because the filling process has this very convenient interpretation as a sum of independent random variables, it is a more natural object to study when considering asymptotics that are more refined than just large deviation properties. Hence in this paper we will first focus on getting the refined large deviation asymptotics for the filling process.

Throughout this paper the same asymptotic scaling as in [7] will be used. Let $\lfloor a \rfloor$ denote the integer part of $a \in [0, \infty)$. Fix $\theta \in (0, \infty)$, let $r = \lfloor \theta n \rfloor$ and consider the limit as $n \to \infty$. Suppose that $\Gamma_0^n(\theta)$ denotes the fraction of urns that are empty after all r balls have been thrown. It is easy to derive a law of large numbers limit, and indeed the fraction of unoccupied urns converges in probability to $e^{-\theta}$ by the well known Poisson approximation [14]. In this context, $1-\Gamma_0^n(\theta) \geq \xi > 1-e^{-\theta}$ corresponds to a rare event (exceptionally many occupied urns), and likewise $1-\Gamma_0^n(\theta) \leq \xi < 1-e^{-\theta}$ constitutes a rare event (exceptionally many empty urns). Our main results, which are stated in Section 2, are explicit higher order asymptotic approximations for the probabilities of such rare events.

For classical occupancy models, combinatorial formulas for certain probabilities can be obtained using the inclusion-exclusion principle. However, since in the setting of rare event problems one must add and subtract quantities that are large compared to the quantity one hopes to compute, these "exact" formulas are not always useful. Indeed, for some of the calculations given at the end of the paper the exact formulas did not give a meaningful answer. Also, because they are not analytic expressions the formulas do not provide much in the way of qualitative insight. Asymptotic approximations, and in particular large deviation approximations, often provide a more useful alternative.

The classical occupancy problem arises in several applications. For example, in complexity theory it appears in connection with the random 3-SAT problem [13]. Here there are n variables, and formulas consisting of cn boolean clauses with 3 distinct variables are chosen at random according to a uniform distribution. If c is too large, no value of the variables will satisfy the formula, i.e., make its value true, with high probability. In [13] it is shown that if $c > 4.76$ this is the case by using a large deviations analysis of the classical occupancy model originated by Weiss [19]. In this model the variables are taken as the urns and a certain type of clause (one with exactly one unnegated variable) are taken as the balls.

In connection with statistical hypothesis testing, large deviation approximations (and in particular refined approximations such as those we describe later on) can be used to construct confidence intervals for tail probabilities for which the central limit theorem might give poor estimates. For example, in [5] the problem is to determine how many sensors are active in a network. One keeps a count of those which have responded to query signals. Suppose each response confirms the activity of one randomly chosen sensor and it is desired to estimate the number of the active sensors with high probability. Estimators, confidence intervals, and estimates for error probabilities can all be constructed for networks with a hundred or more (active) nodes using the refined large deviation asymptotics described here.

As a final example large deviation approximations for the classical occupancy model can also be used to dimension optical switches. Here the problem is to determine the number of shared any-color to any-color wavelength converters that would be needed to provide satisfactory transmission in a bufferless optical packet switch. Other applications include data bases [1], and more recently coding theory [15].

There is a small collection of papers that study the sort of "higher order" large deviation approximations considered in the present paper. The first work in this area is Bahadur and Rao [3], which considers the sample mean for independent and identically distributed random variables. Iltis [11] considers higher order large deviation asymptotics for Markov-additive random variables in $\mathbb{R}^d$ where the chain is regenerative, aperiodic and time homogenous. In both papers, a refined central limit approximation is applied to the twisted distribution, which in the case of [11] uses techniques similar to those in [12]. Also in both papers the underlying processes are time homogeneous, which differs from the state dependency present in the evolution of the occupancy process.

A paper which proves higher order approximations for processes with state dependency is Azencott [2]. This paper considers solutions of "small noise" stochastic differential equations and gives refined approximations for probabilities of sets of trajectories. These sets of trajectories must satisfy a certain smoothness condition on the boundary of the set. The corresponding central limit approximations are somewhat more straightforward in that the original process is defined in terms of a Gaussian driving noise. Although the rate coefficients are not given in explicit form, the structure of the approximation is similar to one that we obtain for a certain parameter regime in the occupancy problem.

Lastly we note that Fleming and James [10] also investigate higher order asymptotics for the probability that a "small noise" diffusion process exits a fixed domain before a given fixed time. However, the results of [10] apply only to initial conditions for which the large deviation minimal cost trajectory exits before the given terminal time. Although some of the problems we consider could be formulated as such "exit time" problems, it turns out that the minimal cost trajectory will always exit exactly at the terminal time, and so the methods used in [10] do not seem to apply.

In terms of technique, our approach is closest to that of [3]. However, the final form of the result is qualitatively quite different from that of [3], and this is due to both the state dependencies and to certain natural boundaries on the state space of the occupancy process (one cannot have more filled urns than the total number of urns available).

In what follows we omit the case where the urns are not all empty initially, which requires only minor modifications of the methods that we use. We expect that these methods would also apply with only small adjustments to the case when balls "miss the urns" with a fixed probability $p > 0$ independently at each trial [18], but have not verified all details at the present time. A more substantial generalization is to consider statistics other than Maxwell-Boltzmann, such as Bose-Einstein or Fermi-Dirac. Another interesting generalization is refined large deviation approximations for the random vector whose ith component is the fraction of urns containing exactly i balls, $i \leq I$ for a fixed constant I.

The rest of the paper is organized as follows. In Section 2 we define the probability model and review known large deviation results for the classical occupancy problem. These include a description of an importance sampling scheme which allows accurate empirical estimation of rare event probabilities in classical occupancy, which will be used later when we present data on these various approximations. We then state the main results. The proof is given in Section 3. It relies on a refined central limit approximation for sums of independent but non-identically distributed random variables lying on a common lattice, and generalizes several results in [9, Chapter XVI]. Finally, in Section 4 we present some numerical results for the asymptotics as well as approximate values obtained using an "exact" approach and the importance sampling scheme.

2 Review and Main Results

2.1 Large Deviations for the Occupancy Process

We restrict attention to occupancy models with Maxwell-Boltzmann statistics. In [7, 4] a large deviations principle was proved occupancy models with the number of urns n as the scale parameter and the number of balls $r = \lfloor n\theta \rfloor$ in fixed proportion as $n \to \infty$. [7] includes a fairly complete solution to the associated calculus of variations problem that must be solved to obtain the large deviations exponent. In what follows we restrict to the case of starting with all urns initially empty, the case where some urns are already occupied being a straightforward extension. Suppose that a total of i balls (or $t = i/n$ balls per urn) have been thrown at some stage in the experiment. Then the fraction of occupied urns $\Gamma_{0+}^n(t)$ performs a random walk, with $\Gamma_{0+}^n(0) = 0$ and

$$\begin{aligned} \Gamma_{0+}^n\left(t + \frac{1}{n}\right) &= \Gamma_{0+}^n(t), \qquad w.p.\ \Gamma_{0+}^n(t) \\ \Gamma_{0+}^n\left(t + \frac{1}{n}\right) &= \Gamma_{0+}^n(t) + \frac{1}{n}, \text{ otherwise.} \end{aligned}$$

By the well known Poisson approximation [14], after $n\theta$ balls have been thrown (or θ balls per urn), the fraction of empty urns is approximately $e^{-\theta}$. Thus there are two different rare events of interest:

$$\begin{aligned} \mathcal{V}_1 &\doteq \{\Gamma_{0+}^n(\theta) \geq \xi : \xi < \theta, \xi > 1 - e^{-\theta}\}, \\ \mathcal{V}_2 &\doteq \{\Gamma_{0+}^n(\theta) \leq \xi : \xi < 1 - e^{-\theta}\}. \end{aligned} \tag{2.1}$$

$\mathcal{V}_1$ corresponds to the rare event that exceptionally many urns are occupied and $\mathcal{V}_2$ corresponds to the rare event that exceptionally few are occupied. In what follows we assume that $n\xi$ and $n\theta$ are both nonnegative integers.

Remark 2.1. *Note that in the case of event $\mathcal{V}_2$ $\theta > \log(1-\xi) > \xi$. In the case of event $\mathcal{V}_1$ we impose the condition $\xi < \theta$, but there is no real loss of generality. If $\xi > \theta$ then we need more nonempty urns than there are balls, which is impossible (i.e., $P\{\Gamma_{0+}^n(\theta) \geq \xi\} = 0$). If $\xi = \theta$ then $\{\Gamma_{0+}^n(\theta) \geq \xi\}$ corresponds to the event that every ball falls into an empty urn, an event whose probability is $\frac{(n-n\theta)!}{n!}$. Note also that when $\xi = 1$, $\mathcal{V}_1$ is a rare event for any $1 < \theta < \infty$, and that there is no $\mathcal{V}_2$ type rare event. In this case, the event $\mathcal{V}_1$ means that every urn is occupied after $n\theta$ balls have been thrown.*

Let $\mathcal{Q}(\mathbb{Z}_+)$ denote the set of probability measures on the non-negative integers, and define the *relative entropy* of $\mu \in \mathcal{Q}(\mathbb{Z}_+)$ with respect to $\nu \in \mathcal{Q}(\mathbb{Z}_+)$ by

$$D\left(\mu \,||\, \nu\right) \doteq \sum_{i \in \ +} \log\left(\mu_i/\nu_i\right) \mu_i$$

(with the understanding that $0 \log 0 = 0$). The large deviations exponent for both the events described above is given by (cf. [7, 16])

$$J(\theta) \doteq \inf_{\Gamma} D\left(\Gamma \,||\, \mathcal{P}(\theta)\right),$$

where the infimum is over all $\Gamma \in \mathcal{Q}(\mathbb{Z}_+)$ subject to $\Gamma_0 = 1 - \xi$ and $\sum_i i\Gamma_i = \theta$, and $\mathcal{P}(\theta)$ denotes the Poisson distribution with parameter θ. The infimum is readily found to be

$$J(\theta) = (\theta - \xi)\log\rho + (1-\xi)\log(1-\xi) - \frac{(1-\rho\xi)}{\rho}\log(1-\rho\xi), \tag{2.2}$$

where ρ is the unique positive root to

$$\theta\rho = -\log(1-\rho\xi)\,. \tag{2.3}$$

To see that such a root exists and is unique, first note that 0 is always a root. The right hand side of (2.3) is a strictly convex function, with derivative with respect to ρ equal to ξ at $\rho = 0$, and which increases to ∞ as $\rho \uparrow \xi^{-1}$. Hence a unique positive solution to (2.3) always exists provided $\theta > \xi$. Moreover, when it does exist, $\rho\xi < 1$.

As is well known in large deviation theory, the asymptotically most likely trajectory for the random walk, conditioned on the outcome of either experiment (for example $\mathcal{V}_1$), can be identified as the "cheapest cost trajectory" for an associated calculus of variations problem. In the present context this is just the absolutely continuous trajectory ψ_0 that minimizes

$$\int_0^\theta D\left(\left(-\dot{\psi}_0(t), 1+\dot{\psi}_0(t)\right) \,||\, (\psi_0(t), 1-\psi_0(t))\right) dt$$

subject to the constraints $\psi_0(0) = 1$ and $\psi_0(\theta) \le 1-\xi$, and with D now relative entropy for probability measures on $\{0,1\}$. As the results in [7] show, the minimizer is just

$$\psi_0(t) = \frac{1}{\rho}e^{-t\rho} + \left(1-\frac{1}{\rho}\right).$$

The parameter ρ is useful in change-of-measure importance sampling [17] to obtain empirical estimates for rare event probabilities in classical occupancy. An efficient scheme can be obtained by multiplying the probability that a ball enters an occupied urn by ρ, i.e., to consider a new measure under which

$$\tilde{\mathbb{P}}\left\{\Gamma^n_{0+}\left(t+\frac{1}{n}\right) = \Gamma^n_{0+}(t)\right\} = \rho\Gamma^n_{0+}(t)\ .$$

ρ can be interpreted as a "twist parameter." When $\rho < 1$ (which is true in the case $\xi > 1-e^{-\theta}$) we make it more likely that balls fall into the empty urns. When $\rho > 1$ (true in the case $\xi < 1-e^{-\theta}$) we make it more likely that balls fall into occupied urns.

We now go over some preliminaries for the filling experiment before stating the main results.

2.2 Filling a Given Fraction ξ of Urns

The random number of balls needed to fill a given fraction ξ of the urns is the sum of $n\xi$ non-identical independent random variables. The ith summand represents the number of balls that are thrown to occupy an empty urn when $i-1$ urns are already occupied. By a *failure probability* we mean the probability that a thrown ball lands in an occupied urn, and so the failure probability at time i is $q_i^n = (i-1)/n$. Suppose X_i^n is the number of balls thrown to occupy another empty urn immediately after $i-1$ urns have become occupied. Defining $\mathcal{G}_k(q) \doteq q^{k-1}p, k = 1, 2, \ldots, p = 1-q$, with $p_i^n = 1 - q_i^n$, we have

$$\begin{aligned} \mathbb{P}\left\{X_i^n = k\right\} &= \mathcal{G}_k(q_i^n), \\ Y^n(\xi) &\doteq \frac{\sum_{i=1}^{n\xi} X_i^n}{n}. \end{aligned}$$

In terms of occupancy process, the number of balls per urn $Y^n(\xi)$ needed to occupy a given fraction $\xi \in (0,1]$ of the urns satisfies

$$Y^n(\xi) = \min\left\{t : \Gamma_{0+}^n(t) \geq \xi\right\}.$$

By the Poisson approximation, $Y^n(\xi) \approx -\log(1-\xi)$ balls per urn are required to occupy $n\xi$ urns. Two different kinds of rare events connected with the filling process are

$$\begin{aligned} \mathcal{W}_1 &\doteq \{Y^n(\xi) \leq \theta : \xi < \theta < -\log(1-\xi)\} \\ &= \{Y^n(\xi) \leq \theta : \xi < \theta, \xi > 1 - e^{-\theta}\} \\ \mathcal{W}_2 &\doteq \{Y^n(\xi) \geq \theta : \theta > -\log(1-\xi)\} \\ &= \{Y^n(\xi) \geq \theta : \xi < 1 - e^{-\theta}\}. \end{aligned}$$

Note that when $\xi = 1$ $\mathcal{W}_1$ is a rare event for any $1 < \theta < \infty$ (and there is no type $\mathcal{W}_2$ rare event). $\mathcal{W}_1$ in this case should be interpreted as using no more than θ balls per urn to occupy *all* the urns.

Clearly $\mathcal{V}_1,\ \mathcal{V}_2$ as defined in (2.1) are closely related to $\mathcal{W}_1,\ \mathcal{W}_2$. In fact

$$\mathcal{V}_1 = \mathcal{W}_1. \tag{2.4}$$

However $\mathcal{V}_2 \neq \mathcal{W}_2$. Suppose that $\mathcal{W}_2$ has occurred. Then at least θ balls per urn were required to fill $n\xi$ urns, which implies after throwing θ balls per urn, there must be at most $n\xi$ urns occupied, i.e.,

$$\Gamma_{0+}^n(\theta) \leq \xi,$$

so $\mathcal{V}_2$ has occurred.

However, $\mathcal{V}_2$ can occur even when $\mathcal{W}_2$ does not occur. Suppose that $n\xi$ urns are occupied *strictly before* $n\theta$ balls are thrown, but that all additional balls (up till and including when ball $n\theta$ is thrown) fall into *already occupied* urns. Then $\mathcal{V}_2$ occurs, though $\mathcal{W}_2$ does not. We can formulate this as

$$\mathcal{V}_2 = \mathcal{W}_2 \cup \left\{ Y^n(\xi) < \theta \text{ and } X^n_{n\xi+1} > n\theta - nY^n(\xi) \right\}. \tag{2.5}$$

Let $F_n(\alpha) \doteq \frac{1}{n} \log \mathbb{E}\left[e^{n\alpha Y^n(\xi)}\right]$ be the scaled logmoment generating function. Then one can readily compute

$$F_n(\alpha) = \alpha\xi + \frac{1}{n}\sum_{j=1}^{n\xi} \log(p^n_j) - \frac{1}{n}\sum_{j=1}^{n\xi} \log(1 - q^n_j e^\alpha). \tag{2.6}$$

Fix $\alpha < -\log \xi$. Interpreting the last display as a Riemann sum, letting $n \to \infty$, and then evaluating the resulting integral shows that $F_n(\alpha) \to F(\alpha)$, where

$$F(\alpha) = \alpha\xi - (1-\xi)\log(1-\xi) + \frac{1 - e^\alpha \xi}{e^\alpha} \log(1 - e^\alpha \xi). \tag{2.7}$$

Note that the convergence of F_n to F is true for all values of ξ, including $\xi = 1$. When $\xi = 1$ we restrict to $\alpha < 0$, and $F(\alpha) = \alpha + \frac{1-e^\alpha}{e^\alpha} \log(1 - e^\alpha)$.

Define the function $J_n(\theta)$ to be the Legendre transform of the log moment generating function:

$$J_n(\theta) = \sup_\alpha \{\alpha\theta - F_n(\alpha)\}. \tag{2.8}$$

Direct calculations show that (i) F_n is strictly convex on its domain of finiteness, (ii) $F'_n(-\infty) = \xi$, and (iii) that $F'_n(\alpha) \uparrow \infty$ as $\alpha \uparrow -\log[(n\xi-1)/n]$. It then follows that the supremum in the last display is finite and attained at some unique $\alpha^*_n \leq -\log[(n\xi - 1)/n]$ whenever $\theta > \xi$, with $\theta = F'_n(\alpha^*_n)$. Also observe that $F'_n(0) \approx -\log(1-\xi)$. Hence for sufficiently large n, if $\xi > 1 - e^{-\theta}$ then $\alpha^*_n < 0$ and if $\xi < 1 - e^{-\theta}$ then $\alpha^*_n > 0$. Define

$$\rho_n = \exp \alpha^*_n. \tag{2.9}$$

Then $\theta = F'_n(\alpha^*_n)$ becomes

$$\theta = \xi + \frac{1}{n}\sum_{j=1}^{n\xi} \frac{q^n_j \rho_n}{1 - q^n_j \rho_n} = \frac{1}{n}\sum_{j=1}^{n\xi} \frac{1}{1 - q^n_j \rho_n}. \tag{2.10}$$

Define

$$\begin{aligned} \bar{J}(\theta) &\doteq \sup_\alpha \{\alpha\theta - F(\alpha)\} \\ &= \theta \log \rho - F(\log(\rho)), \end{aligned}$$

where $\alpha^* = \log \rho$ uniquely achieves the supremum. Then simple calculation shows that ρ indeed satisfies (2.3). Inserting the expression for F we recover the equation (2.2), and thus $\bar{J} = J$. Again notice that this holds for both cases $\xi = 1, \xi < 1$.

Later on, we will prove that $\rho_n \to \rho$ and $\alpha_n^* \to \alpha^*$. Define the "twisted variance"

$$\sigma_n^2 \doteq F_n''(\alpha_n^*) = \frac{1}{n} \sum_{j=1}^{n\xi} \frac{q_j^n e^{\alpha_n^*}}{\left(1 - q_j^n e^{\alpha_n^*}\right)^2}. \tag{2.11}$$

Again using the standard Riemann integral approximation and the bound $\alpha^* < -\log \xi$, we have $\sigma_n^2 \to \sigma^2$, where

$$\sigma^2 \doteq F''(\alpha^*) = \int_0^\xi \frac{te^{\alpha^*}}{(1 - te^{\alpha^*})^2} dt.$$

Integrating gives

$$\sigma^2 = \frac{\xi}{1 - \rho\xi} - \theta. \tag{2.12}$$

2.3 Main Results

The main results of this paper can now be stated. We will first give the refined asymptotics for the filling process and then, by incorporating the difference (2.5), analogous results for the occupancy process. We first make a definition.

Definition 2.1. *Consider a sequence of numbers $p_n \in [0, 1]$ and $J \in [0, \infty]$ such that*

$$\frac{1}{n} \log p_n \to J.$$

Then $K \in (0, \infty)$ is a ν-prefactor for $\{p_n\}$ if

$$n^\nu p_n e^{nJ} \to K.$$

Theorem 2.1. *If $\xi < 1, \xi < \theta$ and $\xi > 1 - e^{-\theta}$ then $p_n^l = \mathbb{P}\{Y^n(\xi) \leq \theta\} = \mathbb{P}\{\mathcal{W}_1\}$ has a $\frac{1}{2}$-prefactor with*

$$\lim_{n\to\infty} \sqrt{n}\, \mathbb{P}\{Y^n(\xi) \leq \theta\} e^{nJ(\theta)} = \frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{1}{1-\rho}\right) \sqrt{\frac{1-\rho\xi}{1-\xi}}. \tag{2.13}$$

If $\xi < 1$ and $\xi < 1 - e^{-\theta}$ then $p_n^g = \mathbb{P}\{Y^n(\xi) \geq \theta\} = \mathbb{P}\{\mathcal{W}_2\}$ has a $\frac{1}{2}$-prefactor with

$$\lim_{n\to\infty} \sqrt{n}\, \mathbb{P}\{Y^n(\xi) \geq \theta\} e^{nJ(\theta)} = \frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{\rho}{\rho - 1}\right) \sqrt{\frac{1-\rho\xi}{1-\xi}}. \tag{2.14}$$

If $\xi = 1, \xi < \theta$, $p_n^1 = \mathbb{P}\{Y^n(1) \leq \theta\} = \mathbb{P}\{\mathcal{W}_2\}$ has a 0-prefactor with

$$\lim_{n\to\infty} \mathbb{P}\{Y^n(1) \leq \theta\} e^{nJ(\theta)} = \frac{1}{\sqrt{(1-\rho)\,\sigma^2}}. \tag{2.15}$$

In all cases ρ is the unique positive root of

$$\theta = -\frac{1}{\rho}\log(1-\rho\xi)$$

and σ^2 is determined as in (2.12).

Theorem 2.2. *If $\xi < 1,\ \xi < \theta$ and $\xi > 1 - e^{-\theta}$ then $q_n^l = \mathbb{P}\left\{\Gamma_{0+}^n(\theta) \geq \xi\right\} = \mathbb{P}\{\mathcal{V}_1\}$ has a $\frac{1}{2}$-prefactor with*

$$\lim_{n\to\infty} \sqrt{n}\, \mathbb{P}\left\{\Gamma_{0+}^n(\theta) \geq \xi\right\} e^{nJ(\theta)} = \frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{1}{1-\rho}\right) \sqrt{\frac{1-\rho\xi}{1-\xi}}. \tag{2.16}$$

If $\xi < 1$ and $\xi < 1 - e^{-\theta}$ then $q_n^g = \mathbb{P}\left\{\Gamma_{0+}^n(\theta) \leq \xi\right\} = \mathbb{P}\{\mathcal{V}_2\}$ has a $\frac{1}{2}$-prefactor with

$$\lim_{n\to\infty} \sqrt{n}\, \mathbb{P}\left\{\Gamma_{0+}^n(\theta) \leq \xi\right\} e^{nJ(\theta)} = \frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{\rho}{\rho - 1} + \frac{\rho\xi}{1-\rho\xi}\right) \sqrt{\frac{1-\rho\xi}{1-\xi}}. \tag{2.17}$$

If $\xi = 1$, $q_n^1 = \mathbb{P}\left\{\Gamma_{0+}^n(\theta) = 1\right\} = \mathbb{P}\{\mathcal{V}_1\}$ has a 0-prefactor with

$$\lim_{n\to\infty} \mathbb{P}\left\{\Gamma_{0+}^n(\theta) = 1\right\} e^{nJ(\theta)} = \frac{1}{\sqrt{(1-\rho)\,\sigma^2}}. \tag{2.18}$$

In all cases ρ, σ^2 are obtained as in Theorem 2.1.

Owing to (2.4) the $\frac{1}{2}$-prefactors for p_n^l and q_n^l are the same, as are the 0-prefactors for p_n^1 and q_n^1. However, as suggested by (2.5), the $\frac{1}{2}$-prefactor for q_n^g differs from that of p_n^g .

3 Proof of the Main Results

3.1 An Extension of the Central Limit Theorem

In what follows we will need the following theorem, which generalizes [9, Page 540, Theorem 2] to independent, non-identical lattice random variables. Let $\{W_i^n : 1 \leq i \leq n\}$ be a sequence of independent, non-identical, lattice random variables. Let

$$S_n = \sum_{i=1}^{n} W_i^n,$$

and suppose that

$$\mathbb{E}\left[W_i^n\right] = 0, \quad \mathbb{E}\left[(W_i^n)^2\right] = s_i^n, \quad \mathbb{E}\left[(W_i^n)^3\right] = \mu_i^n.$$

Define

$$\sigma_n^2 \doteq \frac{1}{n}\mathbb{E}\left[S_n^2\right] = \frac{1}{n}\sum_{i=1}^{n} s_i^n$$
$$\mu_n \doteq \frac{1}{n}\mathbb{E}\left[S_n^3\right] = \frac{1}{n}\sum_{i=1}^{n} \mu_i^n.$$

Finally, define the characteristic functions $\phi_n(t) \doteq \mathbb{E}\left[e^{itS_n}\right]$ and $\phi_k^n(t) \doteq \mathbb{E}\left[e^{itW_k^n}\right]$.

The theorem requires that the sequence of distributions satisfy the following conditions:

Condition 3.1. *1.* $\lim \sigma_n = \sigma$, *where* $0 < \sigma < \infty$.

2. $\lim \mu_n = \mu$, *where* $\mu \in \mathbb{R}$.

3. *There is* $0 < C < \infty$ *such that for any* n *and all* $1 \leq i \leq n$, $s_i^n \leq C$ *and* $|\mu_i^n| \leq C$.

4. *There is* $0 < C < \infty$ *such that for any* n *and all* $1 \leq i \leq n$, $\mathbb{E}(W_i^n)^4 \leq C$ *(without loss we assume* C *is the same constant as in part 3).*

5. *For any* $\delta > 0$, *there exists* $0 < b < 1$ *so that for all* $t \in [-\pi, -\delta] \cup [\delta, \pi]$ *and all* n,
$$|\phi_n(t)| \leq b^n.$$

Remark 3.1. *Parts 1 to 4 of the condition are mild and easy to check. However part 5 can be nontrivial. It is easy to check when ϕ_n is of the form $[\phi(t)]^n$, corresponding to identically distributed random variables. However, for non-identical random variables it need not hold. We thus give a simple sufficient condition for part 5 which is adequate for the present problem.*

Condition 3.2. *For any $\delta > 0$ there exist constants $c \in (0,1), \zeta \in (0,1)$, and for each $n \in \mathbb{N}$ there exists a subset Λ_n of $\{1, 2, \ldots, n\}$ such that $|\Lambda_n| \geq \zeta n$ and*

$$\left|\phi_j^n(t)\right| \leq c, \text{for all } j \in \Lambda_n \text{ and any } t \in [-\pi, -\delta] \cup [\delta, \pi].$$

It is straightforward to show that Condition 3.2 implies part 5 of Condition 3.1.

The following theorem gives a refined expansion of the central limit theorem for non-identical independent lattice random variables. In [9, Page 531, Chapter XVI], approximation theorems are proved for i.i.d random variables (lattice and non-lattice), and an outline is given for the non-identical, non-lattice case. However, even in the non-identical lattice case the results in [9] take us most of the way, although a replacement for the argument leading to the estimate (4.13) in [9, Page 541] is needed, see our condition below. The proof of the following theorem can be found in [20].

Theorem 3.3. *Suppose $W_i^n, i = 1, \ldots, n$ are independent lattice random variables, with lattice spacing h, first three moments $(0, s_i^n, \mu_i^n)$ and bounded fourth moments satisfying Condition 3.1. Let $H_n(x)$ be the distribution function of*

$$\frac{S_n}{\sqrt{n\sigma_n^2}}$$

where $n\sigma_n^2 \doteq \sum_{i=1}^n s_i^n$ and let $H_n^{\#}(x)$ be the distribution function which is obtained by interpolating $H_n(x)$ linearly through the midpoints of the lattice. Let $\mathcal{N}(x)$ be the cumulative distribution function of the standard $N(0,1)$ distribution and $\mathfrak{N}$ be the corresponding density. Then

$$D_n(x) \doteq H_n^{\#}(x) - \mathcal{N}(x) - \frac{\mu_3^n}{6\sigma_n^3\sqrt{n}}(1-x^2)\mathfrak{N}(x) = o\left(\frac{1}{\sqrt{n}}\right)$$

uniformly in x.

Observe that by construction $H_n(x) = H_n^{\#}(x)$ at midpoints of their lattice, which has spacing $h/\left(\sigma_n\sqrt{n}\right)$.

3.2 A Limit Theorem for the Exponents

Recall the definitions of ρ and ρ_n in Sections 2.1 and 2.2, where ρ is the unique positive root to (2.3) and ρ_n is characterized by (2.10). Our first goal is to show that $\lim_{n\to\infty} n(\rho_n - \rho)$ exists and identify the limit. Recall from the discussion below (2.3) that $\rho\xi < 1$. It is also worth recalling that the assumption that $\theta > \xi$ is harmless, in that $\theta = \xi$ and $\theta < \xi$ correspond to situations that are easy to analyze and vacuous, respectively. See Remark 2.1.

Lemma 3.4. *Assume $\xi \le 1$, $\theta > \xi$, so that ρ_n is well defined by (2.10) and ρ is well defined in (2.3). Then*

$$\rho_n - \rho = \frac{K}{n} + o\left(\frac{1}{n}\right),$$

where

$$K = \frac{1}{2}\frac{\rho^3\xi}{(1-\rho\xi)\log(1-\rho\xi)+\rho\xi}.$$

Proof. Define $f(\rho, x) \doteq \frac{1}{1-\rho x}$ and recall that $q_j^n = (j-1)/n$ for $j = 1, \ldots, n\xi$. From the definition of ρ and using the fact that $n\xi$ is an integer, it follows that

$$\begin{aligned}
\theta &= \int_0^\xi \frac{1}{1-\rho x} dx \\
&= \sum_{j=1}^{n\xi} \int_{q_j^n}^{q_{j+1}^n} f(\rho, x) dx \\
&= \frac{1}{n}\sum_{j=1}^{n\xi} f(\rho, q_j^n) + \sum_{j=1}^{n\xi} \int_{q_j^n}^{q_{j+1}^n} \left(f(\rho, x) - f(\rho, q_j^n)\right) dx.
\end{aligned}$$

By elementary arguments using Taylor's Theorem the last expression is

$$\frac{1}{n}\sum_{j=1}^{n\xi} f(\rho, q_j^n) + \frac{1}{2n^2}\sum_{j=1}^{n\xi} \frac{\partial}{\partial x} f(\rho, q_j^n) + \frac{M_n(\xi,\rho)n\xi}{6n^3},$$

where $M_n(\xi, \rho)$ accounts for the remainder terms and $M = \sup_n M_n(\xi, \rho) < \infty$. Since (2.10) can be expressed as

$$\theta = \frac{1}{n}\sum_{j=1}^{n\xi} f(\rho_n, q_j^n),$$

we obtain

$$\frac{1}{n}\sum_{j=1}^{n\xi}(f(\rho,q_j^n)-f(\rho_n,q_j^n))+\frac{1}{2n^2}\sum_{j=1}^{n\xi}\frac{\partial}{\partial x}f(\rho,q_j^n)+O\left(\frac{1}{n^2}\right)=0. \quad (3.1)$$

We pause to estimate the difference $\rho_n-\rho$. Recall that $\rho\xi<1$. Fix $\bar\rho>\rho$ so that $\bar\rho\xi<1$, and denote

$$I(\alpha)\doteq\int_0^\xi f(\alpha,x)dx.$$

Then $I(\rho)=\theta$. Let the solution for $I(\alpha)=\tilde\theta$ be denoted $\rho_{\tilde\theta}$, and observe that $I(\alpha)$ is monotone increasing and continuous in α. Thus for sufficiently small $\epsilon>0$, $\rho_{\theta+\epsilon}<\bar\rho$. Let

$$S_n(\alpha)\doteq\frac{1}{n}\sum_{j=1}^{n\xi}f(\alpha,q_j^n).$$

Then $S_n(\rho_n)=\theta$. Also observe that S_n is monotone increasing and continuous in α as well. Since $f(\alpha,x)$ is monotone increasing in x, for any $\rho<\alpha<\bar\rho$

$$S_n(\alpha)<I(\alpha)<S_n(\alpha)+\frac{1}{n}\frac{1}{1-\xi\alpha}, \quad (3.2)$$

where the first inequality uses that $S_n(\alpha)$ is a lower Riemann sum, and the second uses that $S_n(\alpha)+\frac{1}{n}\frac{1}{1-\xi\alpha}-\frac{1}{n}$ is an upper Riemann sum. Since (3.2) implies $S_n(\rho)<I(\rho)=\theta$, $\rho_n>\rho$. Inserting $\alpha=\rho_{\theta+\epsilon}$ into (3.2) gives

$$S_n(\rho_{\theta+\epsilon})<\theta+\epsilon<S_n(\rho_{\theta+\epsilon})+\frac{1}{n}\frac{1}{1-\xi\rho_{\theta+\epsilon}}.$$

For $\epsilon>0$ small enough that $\rho_{\theta+\epsilon}<\bar\rho$,

$$|\theta+\epsilon-S_n(\rho_{\theta+\epsilon})|<\frac{A_1}{n},$$

where $A_1=\frac{1}{1-\xi\bar\rho}$. It follows then $\limsup\rho_n<\rho_{\theta+\epsilon}$, and since $\epsilon>0$ can be arbitrary small $\limsup\rho_n\le\rho$. We have already shown $\rho_n>\rho$, and thus

$$\lim_{n\to\infty}\rho_n=\rho.$$

We have shown that for all sufficiently large n $\rho<\rho_n<\bar\rho$. Inserting $\alpha=\rho_n$ into (3.2) and using that $S_n(\rho_n)=\theta=I(\rho)$, one obtains

$$\frac{A_1}{n}\ge|I(\rho_n)-I(\rho)|\ge A_3\,|\rho_n-\rho|\,.$$

Here $A_3 > 0$ is a lower bound on the derivative of I with respect to $\tilde{\rho} \in [\rho, \bar{\rho}]$. It follows that $\limsup_{n\to\infty} n\,|\rho - \rho_n| < \infty$. Returning to equation (3.1) and applying a Taylor series expansion with respect to ρ to the first term,

$$-\sum_{j=1}^{n\xi} \frac{\partial}{\partial \rho} f(\rho, q_j^n) \left((\rho_n - \rho) + O(\rho - \rho_n)^2\right) + \frac{1}{2n} \sum_{j=1}^{n\xi} \frac{\partial}{\partial x} f(\rho, q_j^n) + O\left(\frac{1}{n}\right) = 0.$$

Since $\limsup_{n\to\infty} n\,|\rho - \rho_n| < \infty$, we may consider a subsequence such that $\lim_k n_k\,(\rho - \rho_{n_k}) = K$. Dropping the k subscript to simplify the notation, the last display becomes

$$-\sum_{j=1}^{n\xi} \frac{\partial}{\partial \rho} f(\rho, q_j^n) \left(\frac{K}{n} + O\left(\frac{1}{n^2}\right)\right) + \frac{1}{2n} \sum_{j=1}^{n\xi} \frac{\partial}{\partial x} f(\rho, q_j^n) + O\left(\frac{1}{n}\right) = 0. \quad (3.3)$$

We conclude that

$$\lim_{n\to\infty} -\frac{K}{n} \sum_{j=1}^{n\xi} \frac{\partial}{\partial \rho} f(\rho, q_j^n) + \frac{1}{2n} \sum_{j=1}^{n\xi} \frac{\partial}{\partial x} f(\rho, q_j^n) = 0.$$

Again using that $\rho\xi < 1$,

$$\sup_{0\le j\le n\xi} \sup_{x\in(q_j^n, q_{j+1}^n)} \left| \frac{\partial}{\partial \rho} f(\rho, q_j^n) - \frac{\partial}{\partial \rho} f(\rho, x) \right| \to 0,$$

and

$$\sup_{0\le j\le n\xi} \sup_{x\in(q_j^n, q_{j+1}^n)} \left| \frac{\partial}{\partial x} f(\rho, q_j^n) - \frac{\partial}{\partial x} f(\rho, x) \right| \to 0.$$

Hence by the Lebesgue Dominated Convergence Theorem, each of the Riemann sums in (3.3) converges to the corresponding integral, therefore

$$-K \int_0^\xi \frac{\partial}{\partial \rho} f(\rho, x)dx + \frac{1}{2} \int_0^\xi \frac{\partial}{\partial x} f(\rho, x)dx = 0.$$

This is just

$$-K \int_0^\xi \frac{x}{(1-\rho x)^2} dx + \frac{1}{2}\left(\frac{1}{1-\rho\xi} - 1\right) = 0,$$

and computing the integral gives

$$K = \frac{1}{2} \frac{\rho^3 \xi}{(1-\rho\xi)\log(1-\rho\xi) + \rho\xi}.$$

Since this limit is independent of the subsequence the convergence is proved. $\square$

Next we state a result on the on the asymptotics of $J_n(\theta) - J(\theta)$.

Theorem 3.5. *Assume $\xi < 1$ and $\theta > \xi$, define ρ to be the unique positive root of (2.3), and define J_n and J by (2.8) and (2.2), respectively. Then*

$$\lim_{n\to\infty} n(J(\theta) - J_n(\theta)) = \frac{1}{2}\log\left(\frac{1-\rho\xi}{1-\xi}\right).$$

Proof. For $a > 0$ let $\hat{F}(a) = F(\log a)$ and $\hat{F}_n(a) = F_n(\log a)$, where F and F_n are defined in (2.6) and (2.7), respectively. From the definitions of J and J_n,

$$\begin{aligned} J(\theta) &= (\log\rho)\theta - \hat{F}(\rho) \\ J_n(\theta) &= (\log\rho_n)\theta - \hat{F}_n(\rho_n). \end{aligned}$$

It follows that

$$n(J(\theta) - J_n(\theta)) = n(\log\rho - \log\rho_n)\theta - n(\hat{F}(\rho) - \hat{F}_n(\rho_n)). \tag{3.4}$$

By Lemma 3.4

$$\begin{aligned} \log\rho - \log\rho_n &= -(\rho_n - \rho)\cdot\frac{1}{\rho} + o(\rho_n - \rho) \\ &= -\frac{K}{n}\frac{1}{\rho} + o\left(\frac{1}{n}\right). \end{aligned} \tag{3.5}$$

Also

$$\begin{aligned} \hat{F}(\rho_n) - \hat{F}(\rho) &= (\rho_n - \rho)\hat{F}'(\rho) + o(\rho_n - \rho) \qquad (3.6) \\ &= \frac{K}{n}\hat{F}'(\rho) + o\left(\frac{1}{n}\right). \end{aligned}$$

We continue to estimate the difference:

$$\begin{aligned} \hat{F}(\rho_n) - \hat{F}_n(\rho_n) &= -(1-\xi)\log(1-\xi) + \frac{1-\rho_n\xi}{\rho_n}\log(1-\rho_n\xi) \\ &\quad - \frac{1}{n}\sum_{j=1}^{n\xi}\log(p_j^n) + \frac{1}{n}\sum_{j=1}^{n\xi}\log(1-q_j^n\rho_n) \\ &= \int_0^\xi \log(1-x)dx - \int_0^\xi \log(1-\rho_n x)dx \\ &\quad - \frac{1}{n}\sum_{j=1}^{n\xi}\log(1-q_j^n) + \frac{1}{n}\sum_{j=1}^{n\xi}\log(1-\rho_n q_j^n). \end{aligned}$$

With $f(x) = \log(1-x)$ and $g_n(x) = \log(1-\rho_n x)$ the last equation becomes

$$\begin{aligned}
&\hat{F}(\rho_n) - \hat{F}_n(\rho_n) \\
&= \sum_{j=1}^{n\xi} \int_{q_{j-1}^n}^{q_j^n} (f(x) - f(q_j^n))dx - \sum_{j=1}^{n\xi} \int_{q_{j-1}^n}^{q_j^n} (g_n(x) - g_n(q_j^n))dx \\
&= \sum_{j=1}^{n\xi} f'(q_j^n)\frac{1}{2n^2} - \sum_{j=1}^{n\xi} g_n'(q_j^n)\frac{1}{2n^2} + o\left(\frac{1}{n}\right).
\end{aligned}$$

Therefore

$$n(\hat{F}(\rho_n) - \hat{F}_n(\rho_n)) = \frac{1}{2n}\sum_{j=1}^{n\xi} f'(q_j^n) - \frac{1}{2n}\sum_{j=1}^{n\xi} g_n'(q_j^n) + o(1). \tag{3.7}$$

Since $\xi < 1$, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} n(\hat{F}(\rho_n) - \hat{F}_n(\rho_n)) = \frac{1}{2}\int_0^\xi f'(x)dx - \frac{1}{2}\int_0^\xi g'(x)dx$$

i.e.,

$$\lim_{n\to\infty} n(\hat{F}(\rho_n) - \hat{F}_n(\rho_n)) = \frac{1}{2}\log(1-\xi) - \frac{1}{2}\log(1-\rho\xi). \tag{3.8}$$

Now insert (3.8), (3.6) and (3.5) into (3.4) to obtain

$$\lim_{n\to\infty} n(J(\theta) - J_n(\theta)) = -\frac{K}{\rho}\theta - \frac{1}{2}\log(1-\xi) + \frac{1}{2}\log(1-\rho\xi) + K\hat{F}'(\rho).$$

Since $\theta = F'(\log\rho)$ and $F'(\log\rho) = \hat{F}'(\rho)\rho$, therefore $\hat{F}'(\rho) = \theta/\rho$. Thus

$$\lim_{n\to\infty} n(J(\theta) - J_n(\theta)) = \frac{1}{2}\log\left(\frac{1-\rho\xi}{1-\xi}\right).$$

□

Note that the value of K is not used in the proof at all, only the existence of the limit. Also notice that because of the singular behavior of $\log(1-\xi)$ at $\xi = 1$, we will have to separate the case when $\xi = 1$. In fact when $\xi = 1$ we have the following modified version of Theorem 3.5.

Theorem 3.6. *Assume $\xi = 1$ and $\theta > \xi$, define ρ to be the unique positive root of (2.3), and define J_n and J by (2.8) and (2.2), respectively. Then*

$$\lim_{n\to\infty}\left(n(J(\theta) - J_n(\theta)) - \frac{1}{2}\log n\right) = \frac{1}{2}\log(2\pi(1-\rho)).$$

Proof. First observe that (3.4), (3.5), (3.6) and $\hat{F}'(\rho) = \theta/\rho$ still hold. Thus

$$n(J(\theta) - J_n(\theta)) - \frac{1}{2}\log n = n(\log\rho - \log\rho_n)\theta - n\left(\hat{F}(\rho) - \hat{F}_n(\rho_n)\right) - \frac{1}{2}\log n \tag{3.9}$$

and

$$\begin{aligned} \log\rho - \log\rho_n &= -\frac{K}{n}\frac{1}{\rho} + o\left(\frac{1}{n}\right) \\ \hat{F}(\rho_n) - \hat{F}(\rho) &= \frac{K}{n}\frac{\theta}{\rho} + o\left(\frac{1}{n}\right). \end{aligned}$$

The last two displays imply

$$\lim_{n\to\infty}\left(n(\log\rho - \log\rho_n)\theta + n\left(\hat{F}(\rho_n) - \hat{F}(\rho)\right)\right) = 0. \tag{3.10}$$

The only discrepancy occurs when we compute $\hat{F}_n(\rho_n) - \hat{F}(\rho_n)$. Letting $g_n(x) = \log(1 - \rho_n x)$,

$$\begin{aligned} &\hat{F}_n(\rho_n) - \hat{F}(\rho_n) \\ &= \quad -\frac{1}{n}\sum_{j=1}^{n}\log(1 - q_j^n\rho_n) + \frac{1}{n}\sum_{j=1}^{n}\log(p_j^n) - \frac{1-\rho_n}{\rho_n}\log(1-\rho_n) \\ &= \quad \sum_{j=1}^{n} g_n'(q_j^n)\frac{1}{2n^2} + \left(1 + \frac{1}{n}\sum_{j=1}^{n}\log(p_j^n)\right) + o\left(\frac{1}{n}\right). \end{aligned} \tag{3.11}$$

The last equality is because we interpret $-\frac{1-\rho_n}{\rho_n}\log(1-\rho_n)$ as $\int_0^1 g_n(x)dx + 1$ and then use a Taylor expansion on $g_n(x)$ as was done to obtain (3.7). Since $p_j^n = (n - j + 1)/n, j = 1, \ldots, n$,

$$\frac{1}{n}\sum_{j=1}^{n}\log(p_j^n) = \frac{1}{n}\log\left(\frac{n!}{n^n}\right).$$

By Stirling's formula, [8, Page 54, (9.15)]

$$\frac{1}{n}\left(\log\sqrt{2\pi n} + \frac{1}{(12n+1)}\right) < \frac{1}{n}\sum_{j=1}^{n}\log(p_j^n) + 1 < \frac{1}{n}\left(\log\sqrt{2\pi n} + \frac{1}{12n}\right),$$

and thus

$$\lim_{n\to\infty}\left(n\left(\frac{1}{n}\sum_{j=1}^{n}\log p_j^n+1\right)-\frac{1}{2}\log n\right)=\frac{1}{2}\log(2\pi). \tag{3.12}$$

By the Dominated Convergence Theorem

$$\lim_{n\to\infty}\sum_{j=1}^{n} g_n'(q_j^n)\frac{1}{2n}=\frac{1}{2}\left(g(1)-g(0)\right)=\frac{1}{2}\log(1-\rho). \tag{3.13}$$

Now combining (3.12) and (3.13) with (3.11) gives

$$\lim_{n\to\infty}\left(n\left(\hat{F}_n(\rho_n)-\hat{F}(\rho_n)\right)-\frac{1}{2}\log n\right)=\frac{1}{2}\log(2\pi(1-\rho)).$$

The argument is completed by combining the last display, (3.10), and (3.9). □

3.3 Proofs of the Main Theorems

In this subsection we give the proofs of Theorems 2.1 and 2.2. For reasons outlined in the introduction, we start with Theorem 2.1. Following [3], we first represent the probabilities using a change of measure suggested by the large deviations analysis. This will exhibit each probability as the product of an exponential and an integral. The exponential represents the difference between the large deviation approximation and an exponential term coming from our particular change of measure, and can be approximated using Theorem 3.5. To approximate the integral we use the refined CLT stated in Theorem 3.3, and the final result just combines these two approximations.

Proof of Theorem 2.1, (2.13), (2.14). Recall that $\rho\xi < 1$ and $\rho_n \to \rho$. Thus for sufficiently large n the independent random variables

$$\tilde{X}_j^n \sim \mathcal{G}(\rho_n q_j^n)$$

are well defined. Let

$$\theta_j^n = \mathbb{E}\left[\tilde{X}_j^n\right] = \frac{1}{1-\rho_n q_j^n}$$

and let $\tilde{Z}_j^n \doteq \tilde{X}_j^n - \theta_j^n$ be the corresponding random variables centered at 0. Recalling (2.10) and (2.11), we find that

$$U_n \doteq \frac{\sum_{j=1}^{n\xi}\tilde{X}_j^n - n\theta}{\sqrt{n\sigma_n^2}} \tag{3.14}$$

has 0 mean and unit variance. Let $H_n(u) \doteq \mathbb{P}\{U_n \leq u\}$. U_n is a lattice random variable with lattice points

$$\frac{n\xi - n\theta}{\sqrt{n\sigma_n^2}}, \frac{n\xi - n\theta + 1}{\sqrt{n\sigma_n^2}}, \frac{n\xi - n\theta + 2}{\sqrt{n\sigma_n^2}}, \ldots.$$

Let

$$d_n \doteq \frac{1}{\sqrt{n\sigma_n^2}}$$

denote the lattice step size, and observe that 0 is a lattice point because $n\theta, n\xi$ are integers and $\theta > \xi$.

By expressing $\mathbb{P}\{Y^n(\xi) \leq \theta\}$ and $\mathbb{P}\{Y^n(\xi) \geq \theta\}$ in terms of U_n via the change of measure that relates the distribution of X_j^n to that of $\tilde{X}_j^n$, we obtain

$$\mathbb{P}\{Y^n(\xi) \leq \theta\} = e^{-nJ_n(\theta)} \int_{\{u \leq 0\}} \exp(-\alpha_n^* u \sqrt{n} \sigma_n) dH_n(u) \tag{3.15}$$

$$\mathbb{P}\{Y^n(\xi) \geq \theta\} = e^{-nJ_n(\theta)} \int_{\{u \geq 0\}} \exp(-\alpha_n^* u \sqrt{n} \sigma_n) dH_n(u) \tag{3.16}$$

(see the Appendix for the details of this calculation). Therefore

$$\sqrt{n}\, \mathbb{P}\{Y^n(\xi) \leq \theta\}\, e^{nJ(\theta)} = e^{n(J(\theta) - J_n(\theta))} \int_{\{u \leq 0\}} \sqrt{n}\ \exp(-\alpha_n^* u \sqrt{n} \sigma_n) dH_n(u) \tag{3.17}$$

$$\sqrt{n}\, \mathbb{P}\{Y^n(\xi) \geq \theta\}\, e^{nJ(\theta)} = e^{n(J(\theta) - J_n(\theta))} \int_{\{u \geq 0\}} \sqrt{n}\ \exp(-\alpha_n^* u \sqrt{n} \sigma_n) dH_n(u). \tag{3.18}$$

From now on, let us focus on the proof of (2.13). Denote

$$A_n \doteq n(J(\theta) - J_n(\theta)) \quad \text{and} \quad B_n \doteq \int_{\{u \leq 0\}} \sqrt{n}\ \exp(-\alpha_n^* u \sqrt{n} \sigma_n) dH_n(u). \tag{3.19}$$

First notice that in (2.13) we are in the case $\xi < \theta, \xi > 1 - e^{-\theta}$. As was discussed below (2.8), this implies $\alpha_n^* < 0$. Since $H_n(u)$ is the cumulative distribution function of the lattice variable U_n, the integral in B_n can be written as

$$\begin{aligned} B_n &= \sqrt{n} \sum_{k=-\infty}^{0} \exp(-\alpha_n^* k) \left[H_n(kd_n) - H_n((k-1)d_n)\right] \\ &= \sqrt{n} \left(\sum_{k=-\infty}^{-1} H_n(kd_n) \left[\exp(-\alpha_n^* k) - \exp(-\alpha_n^*(k+1))\right] + H_n(0) \right). \end{aligned}$$

Since $\alpha_n^* < 0$,

$$B_n = \sqrt{n} \sum_{k=-\infty}^{-1} (H_n(kd_n) - H_n(0)) \left[\exp(-\alpha_n^* k) - \exp(-\alpha_n^*(k+1))\right].$$

By definition (2.9) $\rho_n = \exp(\alpha_n^*)$, and therefore

$$\begin{aligned} B_n &= \sqrt{n} \sum_{k=-\infty}^{-1} (H_n(kd_n) - H_n(0)) \left(\rho_n^{-k} - \rho_n^{-(k+1)}\right) \\ &= \sqrt{n} \sum_{k=1}^{\infty} (H_n(0) - H_n(-kd_n)) \left(\rho_n^{k-1} - \rho_n^{k}\right). \end{aligned}$$

Let $\Delta_n \doteq 1/(2\sqrt{n\sigma_n^2})$, and $H_n^{\#}(u)$ be the distribution function obtained from $H_n(u)$ by linear interpolation through the midpoints as stated in Theorem 3.3. Therefore if u is one of the midpoints of the lattice then $H_n^{\#}(u) = H_n(u)$ by construction. Also, since H_n is piecewise constant and jumps just at the lattice points, $H_n(kd_n) = H_n^{\#}(kd_n + \Delta_n)$ for any $k \in \mathbb{Z}$. By setting $u_k^n \doteq kd_n + \Delta_n$ and inserting this into the formula for B_n, one obtains

$$B_n = \sqrt{n} \sum_{k=1}^{\infty} \left(H_n^{\#}(u_0^n) - H_n^{\#}(u_{-k}^n)\right) \left(\rho_n^{k-1} - \rho_n^{k}\right). \tag{3.20}$$

We now apply the normal approximation to $H_n^{\#}(u)$ as given in Theorem 3.3. In the notation of that theorem $W_k^n = \tilde{Z}_k^n, k = 1, \cdots, n\xi$ and $W_k^n = 0$ otherwise. Parts 3 and 4 of Condition 3.1 are satisfied as $\tilde{Z}_k^n$ are lattice random variables with uniformly bounded fourth moments:

$$\mathbb{E}\left[(\tilde{Z}_k^n)^4\right] < C < \infty.$$

This uniform bound follows from the fact that for any $\epsilon \in (0, 1-\rho\xi)$ there is a uniform bound on the failure probabilities $\rho_n q_k^n < \rho\xi + \epsilon < 1$ for all sufficiently large n. Since the $\tilde{Z}_k^n$ have 0 mean,

$$\mu_n \doteq \frac{1}{n} \sum_{k=1}^{n\xi} \mathbb{E}\left[(\tilde{Z}_k^n)^3\right].$$

It is readily verified that

$$\mathbb{E} \sum_{j=1}^{n\xi} \left(\tilde{Z}_j^n\right)^2 = n\sigma_n^2,$$

where σ_n^2 is obtained through (2.11). Moreover $\sigma_n^2 \to \sigma^2 = F''(\alpha^*)$ [see Section 2.2]. It is also readily verified that $\mu_n = F_n'''(\alpha_n^*)$, and in fact $\mu_n \to F'''(\alpha^*)$. Hence parts 1 and 2 of Condition 3.1 are also satisfied. Verification of part 5 follows from Lemma A.1, which is stated and proved in the appendix.

Applying Theorem 3.3 and gives

$$\begin{aligned} & H_n^{\#}(u_0^n) - H_n^{\#}(u_{-k}^n) \\ &= \mathcal{N}\big(u_0^n\big) - \mathcal{N}(u_{-k}^n) \\ &\quad + \frac{\mu_n}{6\sigma_n^3\sqrt{n}}\left[(1-(u_0^n)^2)\mathfrak{N}(u_0^n) - (1-(u_{-k}^n)^2)\mathfrak{N}(u_{-k}^n)\right] + o\left(\frac{1}{\sqrt{n}}\right), \end{aligned}$$

where $o\left(1/\sqrt{n}\right)$ is uniform in $k \in \mathbb{Z}$.

Observe that for fixed $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \sqrt{n}\Big(\mathcal{N}\big(u_0^n\big) - \mathcal{N}(u_{-k}^n)\Big) = \frac{1}{\sqrt{2\pi}}\frac{k}{\sigma}, \tag{3.21}$$

and that the left hand side is dominated by Kk for some $K < \infty$. In addition,

$$\frac{1}{\sqrt{n}}\left|(1-(u_0^n)^2)\mathfrak{N}(u_0^n) - (1-(u_{-k}^n)^2)\mathfrak{N}(u_{-k}^n)\right| \le \frac{Kk^2}{\sqrt{n}}$$

for some $K < \infty$ and without loss we assume it is the same K used to bound the normal distribution. Also, for each fixed $k \in \mathbb{Z}$,

$$\frac{\mu_n}{6\sigma_n^3}\Big((1-(u_0^n)^2)$$

as $n \to \infty$. Finally, observe that $\rho_n \to \rho$ implies for any $k \in \mathbb{N}$,

$$\rho_n^{k-1} - \rho_n^k \to \rho^{k-1} - \rho^k$$

and $\rho_n < 1$ for sufficient large n. By the Dominated Convergence Theorem

$$\begin{aligned} \lim_{n\to\infty} B_n &= \sum_{k=1}^{\infty} \frac{1}{\sqrt{2\pi}} k \frac{1}{\sigma}\left(\rho^{k-1} - \rho^k\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma(1-\rho)}. \end{aligned} \tag{3.22}$$

Applying Theorem 3.5 to A_n in (3.19) and combining the estimate (3.22), we finally have

$$\lim_{n\to\infty} \sqrt{n}\, \mathbb{P}\left\{Y^n(\xi) \le \theta\right\} e^{nJ(\theta)} = \frac{1}{\sqrt{2\pi\sigma^2}}\left(\frac{1}{1-\rho}\right)\sqrt{\frac{1-\rho\xi}{1-\xi}}.$$

We next prove Theorem 2.1, (2.14). The idea and technique of the proof is largely the same as the proof for (2.13). In this case $\xi < 1 - e^{-\theta}$, and as remarked below (2.9), $\rho_n > 1$ for all sufficiently large n. In formula (3.18), we let

$$B_n \doteq \int_{\{u \geq 0\}} \sqrt{n}\ \exp(-\alpha_n^* u \sqrt{n} \sigma_n) dH_n(u).$$

Omitting a few details, this can be rewritten as

$$\begin{aligned} B_n &= \sum_{k=0}^{\infty} \sqrt{n} \big(H_n(kd_n) - H_n(-d_n) \big) \big(\rho_n^{-k} - \rho_n^{-(k+1)} \big) \\ &= \sum_{k=0}^{\infty} \sqrt{n} \big(H_n^{\#}(u_k^n) - H_n^{\#}(u_{-1}^n) \big) \big(\rho_n^{-k} - \rho_n^{-(k+1)} \big). \end{aligned}$$

Applying Theorem 3.3 in a similar manner as in the last case, the Dominated Convergence Theorem gives

$$\lim_{n \to \infty} B_n = \sum_{k=0}^{\infty} \frac{k+1}{\sqrt{2\pi}\sigma} \big(\rho^{-k} - \rho^{-(k+1)} \big) = \frac{\rho}{\sqrt{2\pi}\sigma(\rho - 1)}.$$

Applying Theorem 3.5 to the exponential part of (3.18) finishes the proof of Theorem 2.1, (2.14). $\square$

Before analyzing the final case $\xi = 1$ for the filling process, we prove the refined asymptotics for the occupancy process when $\xi < 1$.

Proof of Theorem 2.2, (2.16), (2.17). Since (2.4) holds, (2.13) implies (2.16). To analyze (2.17) we use (2.5), which states that $\mathcal{V}_2$ is the union of $\mathcal{W}_2$ and

$$C^n \doteq \left\{ Y^n(\xi) < \theta \text{ and } X_{n\xi+1}^n > n\theta - nY^n(\xi) \right\}.$$

From (2.14) we already know that $\mathbb{P}\{W_2\}$ has $\frac{1}{2}$-prefactor

$$\frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{\rho}{\rho - 1} \right) \sqrt{\frac{1 - \rho\xi}{1 - \xi}}.$$

Thus to show (2.17) we must prove that $\mathbb{P}\{C^n\}$ has $\frac{1}{2}$-prefactor

$$\frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{\rho\xi}{1 - \rho\xi} \right) \sqrt{\frac{1 - \rho\xi}{1 - \xi}}$$

with the same exponent $J(\theta)$. From the definition of $X^n_{n\xi+1}$, it is easy to verify

$$\mathbb{P}\left\{X^n_{n\xi+1} > k\right\} = \xi^k.$$

With $G(x) \doteq \mathbb{P}\{Y^n(\xi) \le x\}$, we can represent $\mathbb{P}\{C^n\}$ as

$$\mathbb{P}\{C^n\} = \int_{\{x<\theta\}} \xi^{n\theta - nx} dG(x).$$

By the same change of measure argument as used in (3.17), and with H_n the cumulative distribution function of U_n defined in (3.14)

$$\begin{aligned}\mathbb{P}\{C^n\} &= e^{-nJ_n(\theta)} \int_{\{u<0\}} \xi^{-u\sqrt{n}\sigma_n} e^{-u\sqrt{n}\sigma_n \alpha_n^*} dH_n(u) \\ &= e^{-nJ_n(\theta)} \int_{\{u<0\}} (\rho_n \xi)^{-u\sqrt{n}\sigma_n}\, dH_n(u). \end{aligned} \tag{3.23}$$

Replacing $\exp \alpha_n^*$ by ρ_n, in (3.22) we showed that

$$\lim_{n\to\infty} \sqrt{n} \int_{\{u\le 0\}} (\rho_n)^{-u\sqrt{n}\sigma_n}\, dH_n(u) = \frac{1}{\sqrt{2\pi}\sigma(1-\rho)}.$$

This limit continues to hold if ρ_n and ρ are replaced by $\rho_n\xi$ and $\rho\xi$, so long as $\rho\xi < 1$. Therefore

$$\begin{aligned}&\lim_{n\to\infty} \sqrt{n} \int_{\{u<0\}} (\rho_n \xi)^{-u\sqrt{n}\sigma_n}\, dH_n(u) \\ &= \quad \frac{1}{\sqrt{2\pi}\sigma(1-\rho\xi)} - \lim_{n\to\infty} \sqrt{n}\, \big(H_n(0) - H_n(u^n_{-1})\big),\end{aligned}$$

where u^n_{-1} is defined in (3.20). Using (3.21) and the refined normal approximation, Theorem 3.3, we have

$$\lim_{n\to\infty} \sqrt{n}\big(H_n(0) - H_n(u^n_{-1})\big) = \frac{1}{\sqrt{2\pi}\sigma}.$$

This shows $\lim_{n\to\infty} \sqrt{n} \int_{\{u<0\}} (\rho_n\xi)^{-u\sqrt{n}\sigma_n}\, dH_n(u) = \frac{\rho\xi}{\sqrt{2\pi}\sigma(1-\rho\xi)}$. Applying Theorem 3.5 to the exponential part of (3.23) shows that $\mathbb{P}\{C^n\}$ has $\frac{1}{2}$-prefactor $\frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{\rho\xi}{1-\rho\xi}\right) \sqrt{\frac{1-\rho\xi}{1-\xi}}$ with the same exponent $J(\theta)$, and hence finishes the proof of Theorem 2.2, (2.17). $\square$

Finally, we will prove the $\xi = 1$ case in Theorem 2.1 and Theorem 2.2, i.e., (2.15), (2.18). Because of (2.4) we need only prove (2.15) and (2.18) will follow.

Proof of Theorem 2.1, (2.15). The only difference between this case and the proof of (2.13) is that we can no longer use Theorem 3.5 to calculate the limit of A_n. We rewrite the representation for $\mathbb{P}\{Y^n(1) \leq \theta\}$ slightly as

$$\mathbb{P}\{Y^n(1) \leq \theta\} e^{nJ(\theta)} = e^{n(J(\theta)-J_n(\theta))} \int_{\{u\leq 0\}} \exp(-\alpha_n^* u\sqrt{n}\sigma_n) dH_n(u),$$

and observe that

$$\lim_{n\to\infty} \int_{\{u\leq 0\}} \sqrt{n}\ \exp(-\alpha_n^* u\sqrt{n}\sigma_n) dH_n(u) = \frac{1}{\sqrt{2\pi}\sigma(1-\rho)}$$

is still valid. By Theorem 3.6

$$\lim_{n\to\infty} \frac{e^{n(J(\theta)-J_n(\theta))}}{\sqrt{n}} = \sqrt{2\pi(1-\rho)}.$$

Putting these together gives

$$\lim_{n\to\infty} \mathbb{P}\{Y^n(1) \leq \theta\} e^{nJ(\theta)} = \frac{1}{\sigma\sqrt{1-\rho}},$$

which completes the proof.

□

4 Numerical Results

The following tables and figures compare the refined asymptotic results of Theorems 2.1 and 2.2 with exact results using the inclusion-exclusion principle and numerical estimates obtained by importance sampling. In all cases importance sampling was conducted for 10^5 trials. In the graphs approximations based on unrefined large deviation asymptotics are also presented.

Table 1 presents results for the filling process, case (2.14), for $\xi = 0.5, \theta = 1.2$, and a range of values for the scale parameter.

As the table shows the refined asymptotic tends to overestimate the probability but with a percentage error that decreases as n increases. At $n = 10$ this is well over 50%, however by $n = 20$ it has fallen to under 30% and by $n = 100$ it is only 6%.

In Figure 1 we depict the combinatorial probability again with $n = 100$, $\theta = 1.2$ as above, and with ξ varying. The probability estimated from the large deviations exponent alone is also shown. This overestimates the true value by roughly a factor of 10.

n	Combinatorial	Refined Approx.	Imp. Sample
10	7.55×10^{-3}	1.22×10^{-2}	7.43×10^{-2}
20	9.56×10^{-4}	1.24×10^{-3}	9.56×10^{-4}
50	2.10×10^{-6}	2.35×10^{-6}	2.08×10^{-6}
100	9.73×10^{-11}	1.03×10^{-10}	9.71×10^{-11}
200	2.74×10^{-19}	2.82×10^{-19}	2.68×10^{-19}

Table 1: Estimates for $\mathbb{P}\{Y^n(\xi) \geq \theta\}, \xi = 0.5, \theta = 1.2$.

The combinatorial probabilities were obtained as follows. Recall that $Y^n(\xi) \geq \theta$ is the event that at least $n\theta$ balls are required to fill $n\xi$ urns. We decompose this according to whether or not exactly $n\theta$ balls are required. Thus we can write

$$\{Y^n(\xi) \geq \theta\} = \{Y^n(\xi) > \theta\} \cup A^{n\xi}_{n\theta}$$

where $A^{n\xi}_{n\theta}$ is the event that the $n\xi$th urn to be filled is filled with the $n\theta$th ball that is thrown. Recall the notation that we used in Section 2.1. $\Gamma^n_0(t)$ is the fraction of empty urns after $\lfloor tn \rfloor$ balls are thrown. Setting $r = n\theta$, $m = n(1-\xi)$, the probability of $A^{n\xi}_{n\theta}$ is

$$\mathbb{P}\left\{\Gamma^n_0\left(\frac{r-1}{n}\right) = \frac{m+1}{n}\right\} \cdot \frac{m+1}{n}.$$

Meanwhile the event $\{Y^n(\xi) > \theta\}$ can be written as $\Gamma^n_0\left(\frac{r}{n}\right) > \frac{m}{n}$. The "exact" calculation of the quantities $\mathbb{P}\left\{\Gamma^n_0\left(\frac{r}{n}\right) > \frac{m}{n}\right\}$ and $\mathbb{P}\left\{\Gamma^n_0\left(\frac{r}{n}\right) \geq \frac{m}{n}\right\}$ can then be obtained using the well known method of inclusion and exclusion as described in [8, Chapter II.11] and as denoted there [8, Chapter II, (11.9)] by

$$x_m(r,n) \doteq \mathbb{P}\left\{\Gamma^n_0\left(\frac{r}{n}\right) \geq \frac{m}{n}\right\}. \tag{4.1}$$

Table 2 is for the case when exceptionally few balls are needed to fill a given fraction of urns, which corresponds to (2.13) in Theorem 2.1. The blanks indicate cases when the combinatorial expression, as computed, gave obviously incorrect values which resulted from rounding errors. The pattern of results is similar to that of Table 1 with the refined approximation overestimating the underlying probability but with a decreasing percentage error as n increases.

We now turn to the case of filling all the urns, Theorem 2.1, (2.15). Figure 2 shows results for the probability of filling $n = 50$ urns with $\theta \in$

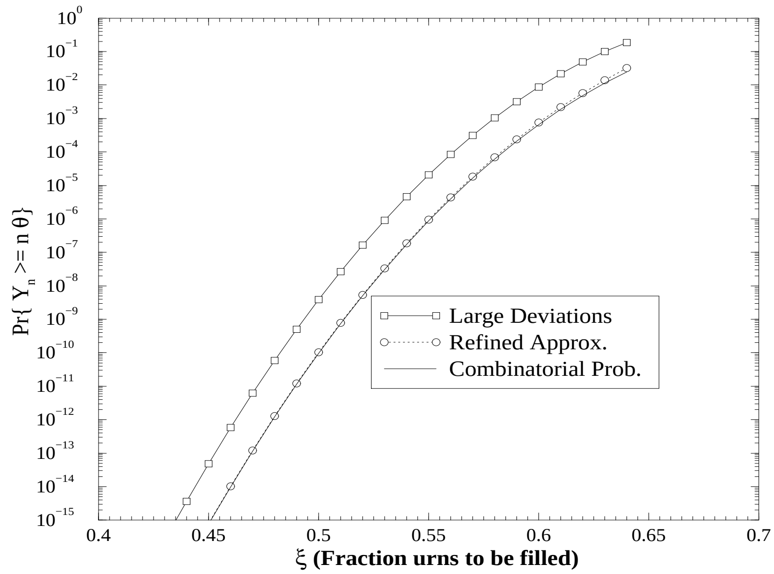

Figure 1: Estimates for $\mathbb{P}\left\{Y^n(\xi) \geq \theta\right\}$ and the "exact" result.

n	Combinatorial	Refined Approx.	Imp. Sample
10	1.53×10^{-1}	1.73×10^{-1}	1.52×10^{-1}
20	2.47×10^{-2}	2.64×10^{-2}	2.46×10^{-2}
50	1.61×10^{-4}	1.68×10^{-4}	1.63×10^{-4}
100	–	5.58×10^{-8}	5.44×10^{-8}
200	–	8.72×10^{-15}	8.65×10^{-15}

Table 2: Estimates for $\mathbb{P}\left\{Y^n(\xi) \leq \theta\right\}, \xi = 0.8, \theta = 1$.

$[1.4, 2.0]$. Sample results are given in Table 3 for the probability of filling urns with only half as many additional balls. Blanks indicate rounding errors in the combinatorial calculation as before.

For Theorem 2.2 we take only the case of (2.17) as the other two results are identical with the corresponding ones for Theorem 2.1. The refined approximation is for the probability that there are at least $(1-\xi)n$ urns empty after θ balls per urn have been thrown. By definition this is determined as $x_m(r, n)$ as we discussed before in (4.1). Furthermore this probability is greater than the probability that it takes $n\theta$ balls to fill $n\xi$ urns as shown in (2.5).

Our results are depicted in Figure 3, again for $n = 100, \theta = 1.2$, as in the

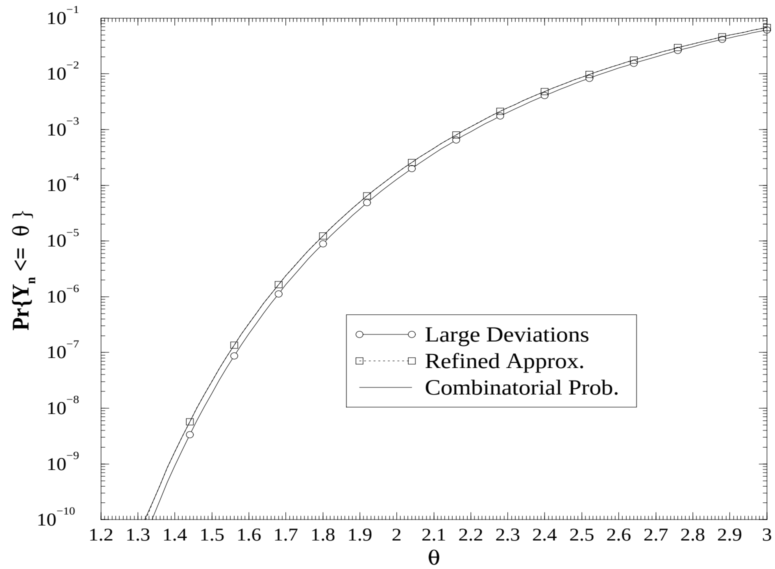

Figure 2: Results for $\mathbb{P}\left\{Y^n(\xi) \leq \theta\right\}$ the refined approximation and "exact" results, $\xi = 1$.

case of Figure 1. These results are given in Table 4. Comparison of Tables 1 and 4 shows that the probabilities differ by a significant factor, e.g., roughly a factor of 4 in the case $n = 50$.

n	Combinatorial	Refined Approx.	Imp. Sample
10	4.60×10^{-2}	4.66×10^{-2}	4.60×10^{-2}
20	1.32×10^{-3}	1.33×10^{-3}	1.32×10^{-3}
50	3.06×10^{-8}	3.07×10^{-8}	3.04×10^{-8}
100	–	5.75×10^{-16}	5.76×10^{-16}
200	–	2.02×10^{-31}	2.02×10^{-31}

Table 3: Estimates for $\mathbb{P}\{Y^n(\xi) \leq \theta\}, \xi = 1.0, \theta = 1.5$.

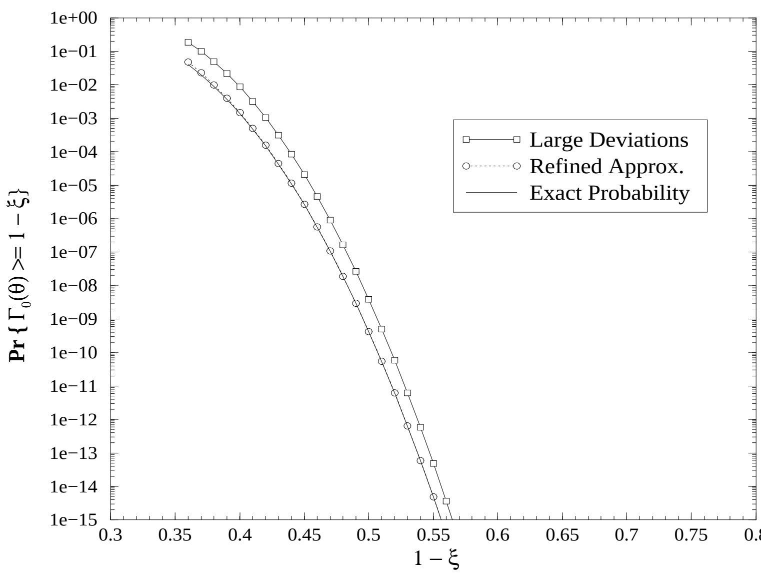

Figure 3: Estimates for $\mathbb{P}\{\Gamma_0^n(\theta) \geq 1 - \xi\}$ and the "exact" result

A Appendix

We begin by showing that Condition 3.2 holds.

Lemma A.1. *The sequences of random variables $\tilde{Z}_j^n$ meet Condition 3.2.*

Proof. In our model

$$Y_n = \sum_{j=1}^{\xi n} \tilde{Z}_j^n,$$

where $\tilde{Z}_j^n = \tilde{X}_j^n - \theta_j^n$ and

$$\tilde{X}_j^n \sim \mathcal{G}(\rho_n q_j^n)$$

n	Combinatorial	Refined Approx.	Imp. Sample
10	4.49×10^{-2}	5.01×10^{-2}	4.49×10^{-2}
20	4.79×10^{-3}	5.11×10^{-3}	4.77×10^{-3}
50	9.39×10^{-6}	9.67×10^{-6}	9.41×10^{-6}
100	4.19×10^{-10}	4.25×10^{-10}	4.18×10^{-10}
200	1.15×10^{-18}	1.16×10^{-18}	1.15×10^{-18}

Table 4: Estimates for $\mathbb{P}\left\{\Gamma_0^n(\theta) \geq 1-\xi\right\}, \xi = 0.5, \theta = 1.2$.

and

$$\theta_j^n = \frac{1}{1-\rho_n q_j^n}.$$

So we can extend the random variables to all $1 \leq j \leq n$ by setting $W_j^n = \tilde{Z}_j^n$ when $j \leq \xi n$ and $W_j^n = 0$ when $\xi n < j \leq n$. We determine $\phi_j^n(t) = \mathbb{E}\left[e^{it\tilde{Z}_j^n}\right]$ for each $\tilde{Z}_j^n$ as

$$\phi_j^n(t) = \frac{e^{i(1-\theta_j^n)t}\rho_n q_j^n}{1-e^{it}\left(1-\rho_n q_j^n\right)} \qquad \text{for } 1 \leq j \leq \xi n.$$

Thus

$$\left|\phi_j^n(t)\right| = \left|\frac{\rho_n q_j^n}{1-e^{it}(1-\rho_n q_j^n)}\right|.$$

Therefore it suffices to show that for all $1 \leq j \leq \xi n$ and $t \in [-\pi, -\delta] \cup [\delta, \pi]$ there is $c > 1$ so that

$$\left|\frac{1-e^{it}(1-\rho_n q_j^n)}{\rho_n q_j^n}\right| > c. \tag{A.1}$$

Let $y_j^n = \frac{1}{\rho_n q_j^n}$, then $y_j^n > \frac{1}{\rho_n \xi}$. Since $\rho_n \to \rho$ and $\rho\xi < 1$ we can assume $y_j^n > y > 1$, where y is some constant. Now we have

$$\left|\frac{1-e^{it}(1-\rho_n q_j^n)}{\rho_n q_j^n}\right|^2 = \left|y_j^n - e^{it}(y_j^n - 1)\right|^2.$$

Define $f(t,x) \doteq |x - e^{it}(x-1)|^2$, so that

$$f(t,x) = x^2 - 2\cos tx(x-1) + (x-1)^2.$$

Since $t \in [-\pi, -\delta] \cup [\delta, \pi]$, for $x > 1$ we have

$$f(t, x) \geq 2(1 - \cos\delta)x^2 - 2(1 - \cos\delta)x + 1 \doteq g(x).$$

$g(x)$ is monotone increasing on $[1/2, \infty)$, and since $y_j^n > y > 1$ we have $g(y_j^n) > g(y)$. Therefore when $t \in [-\pi, -\delta] \cup [\delta, \pi]$,

$$\begin{aligned} \left| \frac{1 - e^{it}(1 - \rho_n q_j^n)}{\rho_n q_j^n} \right|^2 &= f(t, y_j^n) \\ &\geq g(y_j^n) \\ &\geq g(y). \end{aligned}$$

Lastly since $y > 1$ we know $g(y) > g(1) = 1$. Having shown (A.1), it follows that our model satisfies Condition 3.2. □

We next show that the change-of-measure formulas (3.15) and (3.16) are true. Since they are similar, the proof is given for just (3.15).

Proof of (3.15). We recall some previously used and also introduce some new notation:

$$\begin{aligned} G_n(x) &= \mathbb{P}\left\{Y^n(\xi) \leq x\right\} \qquad & G_i^n(x) &\doteq \mathbb{P}\left\{X_i^n \leq x\right\} \\ H_n(x) &= \mathbb{P}\left\{U_n \leq u\right\} \qquad & H_i^n(x) &\doteq \mathbb{P}\left\{\tilde{X}_i^n \leq x\right\}. \end{aligned}$$

Recall also that $\theta_i^n = \mathbb{E}\left[\tilde{X}_i^n\right]$ and that $\sum_{i=1}^{n\xi} \theta_i^n = n\theta$. Denote the left hand side of (3.15) by L and the right hand side by R. In terms of this notation

$$R = e^{-nJ_n(\theta)} \int_{\sum_{i=1}^{n\xi} u_i^n \leq 0} \exp\left(-\alpha_n^* \left(\sum_{i=1}^{n\xi} u_i^n\right)\right) \prod_{i=1}^{n\xi} dH_i^n\left(u_i^n + \theta_i^n\right). \quad \text{(A.2)}$$

Let $d\delta(\cdot)$ be the counting measure on $\mathbb{Z}$. The distribution of each $\tilde{X}_i^n$ has the explicit form

$$dH_i^n(z) = \left(\rho_n q_i^n\right)^{z-1}(1 - \rho_n q_i^n)d\delta(z).$$

Thus

$$
\begin{aligned}
&\prod_{i=1}^{n\xi} dH_i^n\left(u_i^n + \theta_i^n\right) \\
&= \prod_{i=1}^{n\xi} \left(\rho_n q_i^n\right)^{u_i^n + \theta_i^n - 1} \left(1 - \rho_n q_i^n\right) d\delta(u_i^n + \theta_i^n) \\
&= \rho_n^{n\theta - n\xi + \quad {}_{i=1}^{n\xi} u_i^n} \left(\prod_{i=1}^{n\xi} \left[(q_i^n)^{u_i^n + \theta_i^n - 1} \left(1 - \rho_n q_i^n\right) \right] \right) \left(\prod_{i=1}^{n\xi} d\delta(u_i^n + \theta_i^n) \right),
\end{aligned}
\tag{A.3}
$$

where the last equality uses $\sum_{i=1}^{n\xi} \theta_i^n = n\theta$. Since $\rho_n = e^{\alpha_n^*}$

$$
\exp\left(-\alpha_n^* \left(\sum_{i=1}^{n\xi} u_i^n \right) \right) = \rho_n^{- \quad {}_{i=1}^{n\xi} u_i^n}. \tag{A.4}
$$

By definition of $J_n(\theta)$ in (2.8)

$$
\begin{aligned}
e^{-nJ_n(\theta)} &= \exp\left[-n\left(\alpha_n^* \theta - F_n(\alpha_n^*) \right) \right] \\
&= \exp\left(-n\theta\alpha_n^* + n\alpha_n^* \xi + \sum_{i=1}^{n\xi} \log\left(p_i^n\right) - \sum_{i=1}^{n\xi} \log\left(1 - q_i^n e^{\alpha_n^*}\right) \right).
\end{aligned}
$$

Again using $\rho_n = e^{\alpha_n^*}$, this expression can be rewritten as

$$
e^{-nJ_n(\theta)} = \frac{\rho_n^{-n\theta + n\xi} \left(\prod_{i=1}^{n\xi} p_i^n \right)}{\prod_{i=1}^{n\xi} \left(1 - q_i^n \rho_n\right)}. \tag{A.5}
$$

Inserting (A.3), (A.4), and (A.5) into (A.2) gives

$$
R = \int_{\quad {}_{i=1}^{n\xi} u_i^n \le 0} \prod_{i=1}^{n\xi} \left(p_i^n (q_i^n)^{u_i^n + \theta_i^n - 1} d\delta(u_i^n + \theta_i^n) \right).
$$

On the other hand, notice that by definition

$$
L = \int_{\quad {}_{i=1}^{n\xi} x_i^n \le \theta} \prod_{i=1}^{n\xi} dG_i^n(x_i^n).
$$

Using the change of variables $u_i^n = x_i^n - \theta_i^n$,

$$L = \int_{\sum_{i=1}^{n\xi} u_i^n \le 0} \left(\prod_{i=1}^{n\xi} dG_i^n(u_i^n + \theta_i^n) \right).$$

Since G_i^n is the cumulative distribution function of X_i^n

$$L = \int_{\sum_{i=1}^{n\xi} u_i^n \le 0} \prod_{i=1}^{n\xi} \left(p_i^n (q_i^n)^{u_i^n + \theta_i^n - 1} d\delta(u_i^n + \theta_i^n) \right) = R.$$

This completes the proof of (3.15). $\square$

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